Simulating the binary variates for the components of a socio-economical system

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Abstract

Often in practice the components \( W_j \) of a sociological or an economical system \( W \) take discrete 0-1 values. We talk about how to generate arbitrary observations from a binary 0-1 system \( B \) when is known the multidimensional distribution of the discrete random vector \( B \). We also simulated a simplified structure of \( B \) given by the marginal distributions together with the matrix of the correlation coefficients. Different properties of the systems \( W \) are presented too.

Keywords: binary system, marginal distribution, Monte Carlo simulation, random variates, correlation coefficient.

1. Introduction

A general system \( W \) with \( k \) components \( W_1, W_2, W_3, \ldots, W_k \) is characterized by the features \( \lambda_j \) of every variable \( W_j \) and the intensity \( c_{ij} \) of the relation between any two components \( W_i \) and \( W_j \), \( 1 \leq i, j \leq k \). Frequently in practice the relation among the elements of the subsystem \( \{W_i, W_j\} \) is a symmetric one, that is \( c_{ij} = c_{ji} \).

The characteristic \( \lambda_j \) of the component \( W_j \) could be just the parameters which define the marginal distribution of the random variable \( W_j \). In the following we will choose the Pearson correlation coefficient \( \text{Cor}(W_i, W_j) \) to measure the intensity \( c_{ij} \) of the relation which is present between the components \( W_i \) and \( W_j \) of the system \( W \). We mention here that in the literature there are known many other indicators to measure the ratio among the elements \( W_j \) from \( W \) ( [1], [2], [6] ).

Figure 1 presents some kinds of systems \( W \).

Many times in practice the system \( W \) has components \( W_j \) with a normal distribution. Such a system will be designated in the subsequent by \( X \). For this particular case the system components \( X_j \), \( 1 \leq j \leq k \), are dependent normal random variables characterized by their means \( \mu_j \) and their dispersions \( \sigma_j^2 \). So we will take \( \lambda_j = (\mu_j, \sigma_j) \) and \( c_{ij} = \text{Cor}(X_i, X_j) \), \( 1 \leq i, j \leq k \).

Another class from the systems \( W \) are binary 0-1 systems designated by \( B \). The elements \( B_1, B_2, B_3, \ldots, B_k \) of the system \( B \) are binary dependent variables which take only the values 0 and 1. To make a distinction between the systems \( B \) and \( X \) we will use the notation \( r_{ij} = \text{Cor}(B_i, B_j) \) in the discrete case and \( c_{ij} = \text{Cor}(X_i, X_j) \) for the continuous normal marginals variant.

We mention here that the normal type system \( X \) is completely characterized by the set of the parameters \( \mu_i, \sigma_i, c_{ij}, \, 1 \leq i < j \leq k \), that is \( k(k+3)/2 \) values ( [3] ).

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But the multidimensional distribution of an arbitrary binary system $B$ has more parameters. For this reason, in opposition with the normal distributions case, we cannot define a general binary 0-1 system $B$ by knowing only the values $\mu_i, \sigma_i, r_{ij}, 1 \leq i < j \leq k$. More, in the discrete case of $B$, the variance $\sigma^2_j = Var(B_j)$ depends on the mean $\mu_j = Mean(B_j)$. So, knowing only the marginals and the correlation matrix of $B$ we lose a lot of information which define the real multivariate discrete distribution of the system $B$. Some details concerning the behavior of a binary system $B$ will be given in the next section.

We reveal a new other aspect which is present for sociological and economical systems too. So, the individuals of a given population estimate the behavior of each component $W_j$ from a continuous system $W$ by putting subjective marks.

In this approach a binary system $B$ results from $W$ when the marks take only 0 and 1 values. Hence, in practice, we often approximate a continuous system $W$ by a binary one, like $B$. In this case we must evaluate the discretization error.

2. The binary 0-1 systems

The binary random vector $B = (B_1, B_2, B_3, ..., B_k)$ which takes only 0 and 1 values is completely characterized by the probabilities $p_{i_1, i_2, i_3, ..., i_k}$, $i_j \in \{0, 1\}$, $1 \leq j \leq k$, where

$$p_{i_1, i_2, i_3, ..., i_k} = Pr(B_1 = i_1, B_2 = i_2, B_3 = i_3, ..., B_k = i_k)$$

Obviously, $p_{i_1, i_2, i_3, ..., i_k} \geq 0$ for all indices $i_j \in \{0, 1\}$ and in addition

$$\sum_{i_1=0}^{i_1=1} \sum_{i_2=0}^{i_2=1} \sum_{i_3=0}^{i_3=1} ... \sum_{i_k=0}^{i_k=1} p_{i_1, i_2, i_3, ..., i_k} = 1$$

To simplify our expose, for any $i_j \in \{0, 1\}$, we will use the notation

\[\begin{array}{cccc}
\text{Case 1.1. } k = 2 & \text{Case 1.2. } k = 3 \\
\begin{array}{c}
\downarrow \\
W_1 \overset{c_{12}}{\rightarrow} W_2
\end{array} & \begin{array}{c}
\downarrow \\
W_1 \overset{c_{12}}{\rightarrow} W_2 \overset{c_{13}}{\rightarrow} W_3
\end{array} \\
\begin{array}{c}
\downarrow \\
W_2 \overset{c_{12}}{\rightarrow} W_1 \overset{c_{13}}{\rightarrow} W_3
\end{array}
\end{array}\]
So, the equality (1) could be also written in a shorter form as \( p_{+,\ldots,+,} = 1 \).

The marginal distributions of the random vector \( B \) are defined only by the probabilities \( q_j = \Pr(B_j = 1) , \ 1 \leq j \leq k \).

Choosing, for example, the component \( B_1 \) we deduce
\[
\Pr(B_1 = 0) = p_{0,+,\ldots,} = 1 - p_{1,+,\ldots,} = 1 - \Pr(B_1 = 1) = 1 - q_1
\]

**Remark 1.** Since the distribution of the system \( B = (B_1, B_2, \ldots, B_k) \) is determined by the probabilities \( p_{i_1, i_2, \ldots, i_k} \) with the restriction (1) we conclude that a general binary 0-1 system \( B \) with \( k \) components is defined by \( 2^k - 1 \) parameters.

Now we will enumerate some properties of a binary \( B = (B_1, B_2) \) system which has only two components.

We remind that the distribution of an arbitrary 0-1 binary vector \( B = (B_1, B_2) \) is given by the probabilities \( p_{i,j} = \Pr(B_i = i , B_2 = j) \) where \( i, j \in \{0, 1\} \) and \( p_{+,+} = 1 \).

In this case \( q_1 = p_{1,+} = \Pr(B_1 = 1) , \ q_2 = p_{+,1} = \Pr(B_2 = 1) , \ 0 \leq q_1, q_2 \leq 1 \) and therefore
\[
p_{1,0} = q_1 - p_{1,1} , \ p_{0,1} = q_2 - p_{1,1} , \ p_{0,0} = 1 + p_{1,1} - q_1 - q_2
\]

Hence we have the inequalities

**P2.1.** \( \max\{0, q_1 + q_2 - 1\} \leq \min\{q_1, q_2\} \)

After a straightforward calculus we obtain the relations

**P2.2.**
\[
\text{Mean}(B_j) = \text{Mean}(B_j^2) = q_j , \text{Var}(B_j) = q_j(1-q_j) , j \in \{0, 1\}
\]
\[
\eta_2 = \text{Cor}(B_1, B_2) = \frac{p_{1,1} - q_1 q_2}{\sqrt{q_1 (1-q_1)} \sqrt{q_2 (1-q_2)}} , \ 0 < q_1, q_2 < 1
\]

**Remark 2.** This expression of the correlation coefficient \( \eta_2 = \text{Cor}(B_1, B_2) \) does not depend on the concrete values of the binary random variables \( B_1 \) and \( B_2 \). For example, considering \( B_1 \in \{a_1, b_1\} \neq \{0, 1\} , \ B_2 \in \{a_2, b_2\} \neq \{0, 1\} \) we obtain the same value for the indicator \( \eta_2 \).

Since \( q_1 = p_{1,0} + p_{1,1} \) and \( q_2 = p_{0,1} + p_{1,1} \) we prove easily

**P2.3.** If \( p_{1,1} = q_1 q_2 \) then we have also the following equalities
\[
p_{0,0} = (1-q_1) q_2 , \ p_{1,0} = q_1 (1-q_2) , \ p_{0,0} = (1-q_1)(1-q_2)\)

From P2.2 and P2.3 it results

**P2.4.** The binary 0-1 random variables \( B_1, B_2 \) are independent if and only if \( \eta_2 = \text{Cor}(B_1, B_2) = 0 \).

**Remark 3.** The property P2.4 is not always true for an arbitrary continuous two component system \( W = (W_1, W_2) \).

Applying the propositions P2.1 and P2.2 we deduce the inequalities

**P2.5.** \( \text{Cor}(B_1, B_2) \geq \frac{\max\{0, q_1 + q_2 - 1\} - q_1 q_2}{\sqrt{q_1 (1-q_1)} \sqrt{q_2 (1-q_2)}} , \ 0 < q_1, q_2 < 1\)
The following properties are particular cases of the proposition P2.5.

**P2.6.** If \( q_1 = q_2 \) then \( \text{Cor}(B_1, B_2) \leq 1 \)

If \( q_1 = 1 - q_2 \) then \( \text{Cor}(B_1, B_2) \geq -1 \)

Using the formulas

\[
\text{Cov}(1 - B_1, B_2) = -\text{Cov}(B_1, B_2), \quad \text{Var}(1 - B_1, B_2) = \text{Var}(B_1, B_2)
\]

we can prove directly the equalities

**P2.7.** \( \text{Cor}(1 - B_1, B_2) = \text{Cor}(B_1, 1 - B_2) = -\text{Cor}(B_1, B_2) \)

**Graphic 1** presents us a suggestive image of the variation for the lower and upper bounds of \( \eta_2 = \text{Cor}(B_1, B_2) \) index depending on the marginal distributions indicators \( 0 < q_1, q_2 < 1 \).

**Remark 4.** From the propositions P2.1-P2.7 we conclude that the discrete distribution of the system \( B = (B_1, B_2) \) is completely determined by the indices \( 0 < q_1, q_2 < 1 \) which characterize the marginal distributions of \( B \) together with the correlation coefficient \( \eta_2 = \text{Cor}(B_1, B_2), \quad -1 \leq \eta_2 \leq 1 \). But the parameters \( q_1, q_2, \eta_2 \) are mutually dependent (see the properties P2.1 and P2.5 or **Graphic 1**).
3. Generate random observations from a binary system

Leisch, Weingessel and Hornik suggested in [5] the application of the general inverse method for discrete random vectors ([3], [4]) to generate arbitrary observations \((b_1, b_2, b_3, ..., b_k)\), \(b_j \in \{0, 1\}\), for the system \(\mathcal{B} = (B_1, B_2, B_3, ..., B_k)\).

The following algorithm \(GDRV\) produces \((b_1, b_2, b_3, ..., b_k)\) vectors, \(b_j \in \{0, 1\}\), such that

\[
Pr(B_1 = b_1, B_2 = b_2, B_3 = b_3, ..., B_k = b_k) = p_{b_1,b_2,b_3,...,b_k}
\]

where the probabilities \(p_{i_1, i_2, i_3, ..., i_k}, i_j \in \{0, 1\}, 1 \leq j \leq k\), define the binary 0-1 system \(\mathcal{B}\).

**Algorithm GDRV** (Generating Discrete Random Vectors).

Step 0. Input: the probabilities \(p_{i_1, i_2, i_3, ..., i_k}, i_j \in \{0, 1\}, 1 \leq j \leq k\), with \(p_{+,+,+,...,+} = 1\).

Step 1. Establish a one to function \(h: \{1, 2, 3, ..., 2^k\} \rightarrow \{0, 1\}^k\)

Step 2. Compute recurrently the sums

\[
s_0 = 0
s_t = s_{t-1} + p_{h(t)}, \quad 1 \leq t \leq 2^k
\]

Step 3. Generate a random variate \(u\) uniformly distributed on the interval \((0, 1)\)

Step 4. Find the index \(1 \leq t \leq 2^k\) such that \(u \in (s_{t-1}, s_t]\)

Step 5. \(b = h(t)\)

Step 6. Output: \(b\)

Details regarding the theoretical justification of the generating procedure \(GDRV\) can be found in the books [3] and [4].

**Remark 5.** Applying algorithm \(GDRV\) we generated \(n = 10^6\) random variates \((b_1, b_2, b_3)\) from the binary system \(\mathcal{B} = (B_1, B_2, B_3)\) defined by Table 1. For this case the frequencies of the categories \((i_1, i_2, i_3), i_j \in \{0, 1\}, 1 \leq j \leq 3\), are given in Table 2. The validity of the algorithm \(GDRV\) is proved in part since the theoretical values and the empirical estimations of the probabilities \(p_{i_1, i_2, i_3}\) are very closed (compare the results from Tables 1-2).

**Table 1. The theoretical distribution of the binary 0-1 system \(\mathcal{B} = (B_1, B_2, B_3)\)**

<table>
<thead>
<tr>
<th>(P_{0,0,0})</th>
<th>(P_{0,0,1})</th>
<th>(P_{0,1,0})</th>
<th>(P_{0,1,1})</th>
<th>(P_{1,0,0})</th>
<th>(P_{1,0,1})</th>
<th>(P_{1,1,0})</th>
<th>(P_{1,1,1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.050</td>
<td>0.200</td>
<td>0.100</td>
<td>0.150</td>
<td>0.100</td>
<td>0.050</td>
<td>0.050</td>
<td>0.300</td>
</tr>
</tbody>
</table>

**Table 2. The frequencies for the variates \((b_1, b_2, b_3)\) obtained after \(10^6\) simulations with algorithm \(GDRV\)**

<table>
<thead>
<tr>
<th>((0,0,0))</th>
<th>((0,0,1))</th>
<th>((0,1,0))</th>
<th>((0,1,1))</th>
<th>((1,0,0))</th>
<th>((1,0,1))</th>
<th>((1,1,0))</th>
<th>((1,1,1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>49763</td>
<td>200067</td>
<td>99951</td>
<td>149842</td>
<td>99672</td>
<td>49832</td>
<td>50332</td>
<td>300541</td>
</tr>
</tbody>
</table>
4. Systems with normal distributed components

Now we will discuss the case of a system $X = (X_1, X_2, X_3, \ldots, X_k)$ where its components $X_j$, $1 \leq j \leq k$, are random variables with normal distributions.

By $X \sim \text{Norm}(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$, $\sigma > 0$, we understand that the random variable $X$ is normal distributed where $\text{Mean}(X) = \mu$ and $\text{Var}(X) = \sigma^2$. We denote by $\phi(x)$ the Laplace function, that is the cumulative distribution function for the random variable $Z \sim \text{Norm}(0,1)$.

Remind some properties which will be applied in the subsequent.

**P4.1.** If $Z \sim \text{Norm}(0,1)$ and $X = \mu + \sigma Z$ with $\mu \in \mathbb{R}$, $\sigma > 0$ then we have $X \sim \text{Norm}(\mu, \sigma^2)$.

**P4.2** (Inverse method, [3], [4]). If the random variable $U$ is uniformly distributed on the interval $[0,1]$ and $Z = \Phi^{-1}(U)$ then $Z \sim \text{Norm}(0,1)$.

**P4.3.** For any $\mu_i \in \mathbb{R}$, $\sigma_i > 0$, if $X_i \sim \text{Norm}(\mu_i, \sigma_i^2)$ and $Y = X_1 + X_2$ then $Y \sim \text{Norm}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

**Discretization procedure DP.** For any $\alpha \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\sigma > 0$ and $X \sim \text{Norm}(\mu, \sigma^2)$ we designate by $B_{X,\alpha}$ the following binary 0-1 random variable

$$B_{X,\alpha} = \begin{cases} 0, & \text{when } X < \alpha \\ 1, & \text{when } X \geq \alpha \end{cases}$$

Using the procedure DP we deduce by a direct calculus

**P4.4.** For any $X \sim \text{Norm}(\mu, \sigma^2)$ we have $\Pr(B_{X,\alpha} = 1) = 1 - \Phi((\alpha - \mu) / \sigma)$

**P4.5.** For any $-1 \leq c \leq 1$, $Z_j \sim \text{Norm}(0,1)$, the standard normal random variables $Z_1, Z_2$ being independent, if

$$X = Z_1$$

$$Y = cZ_1 + \sqrt{1-c^2}Z_2$$

then $X \sim \text{Norm}(0,1)$, $Y \sim \text{Norm}(0,1)$ and more $\text{Cor}(X,Y) = c$.

**Remark 6.** By using a normal random variable $X \sim \text{Norm}(\mu, \sigma^2)$ and a given bound $\alpha \in \mathbb{R}$ we build a binary 0-1 random variable $B_{X,\alpha}$ such that

$$q = \Pr(B_{X,\alpha} = 1) = 1 - \Phi((\alpha - \mu) / \sigma)$$

(see the discretization procedure DP and Proposition P4.4). When $\mu = 0$ and $\sigma = 1$, the threshold $\alpha \in \mathbb{R}$ determine effectively the distribution of the discrete 0-1 random variable $B_{X,\alpha}$.

5. A discretization process

Having a continuous normal distributed system $X = (X_1, X_2, X_3, \ldots, X_k)$ and fixing some arbitrary thresholds $a_1, a_2, a_3, \ldots, a_k \in \mathbb{R}$ we can obtain a binary 0-1 system $B = (B_1, B_2, B_3, \ldots, B_k)$ with $B_j = B_{X_j,a_j}$, $1 \leq j \leq k$ (apply the procedure DP).

More, when $X_j \sim \text{Norm}(0,1)$, $1 \leq j \leq k$, then $q_j = \Pr(B_j = 1) = 1 - \Phi(a_j)$.
Obviously, in this last case, the correlation indicators \( r_{ij} = \text{Cor}(B_i, B_j) \) and \( c_{ij} = \text{Cor}(X_i, X_j) \), \( 1 \leq i, j \leq k \), have not equal values. More precisely, a correlation coefficient \( r_{ij} \) depends on the quantities \( c_{ij}, q_i, q_j \). The effective relation between \( r_{ij} \) and \( c_{ij} \) indices will be established in the subsequent by applying a stochastic Monte Carlo simulation.

**Remark 7.** For an arbitrary \(-1 \leq c \leq 1\), propositions P4.2 and P4.5 permit us to generate two dependent standard normal random variables \( X, Y \) having just the Pearson correlation coefficient \( \text{Cor}(X, Y) = c \). We can apply Proposition P4.2 (the inverse method, [3], [4]) to generate independent \( Z_i \sim \text{Norm}(0,1) \) random variables which are used by Proposition P4.5.

Now, keeping all the previous notations, we will suggest a Monte Carlo procedure \( \text{MCRCC} \) to establish the real ratios between the correlation coefficients \( c_{ij} = \text{Cor}(X_i, X_j) \) and \( r_{ij} = \text{Cor}(B_i, B_j) \).

**Procedure MCRCC.**

**Step 1.** We generate random variates of volume \( n \) for a bidimensional random vector \((X_1, X_2)\) with standard normal dependent marginals and \( c_{12} = \text{Cor}(X_1, X_2) \), \(-1 \leq c_{12} \leq 1\) (more details in Remark 7).

**Step 2.** Knowing the marginal probabilities \(-1 \leq q_1, q_2 \leq 1\), we specify the discretization thresholds, that is \( a_1 = \Phi^{-1}(1 - q_1) \), \( a_2 = \Phi^{-1}(1 - q_2) \).

**Step 3.** We obtain 0-1 binary samples \((b_1, b_2)\) from the random vector \( B = (B_i, B_j) \) considering the discretization procedure \( B_1 = B_{X_1, a_1}, B_2 = B_{X_2, a_2} \) (algorithm DP).

**Step 4.** Using the samples resulted for \( B = (B_i, B_j) \) we estimate the correlation coefficient \( \eta_{12} = \text{Cor}(B_1, B_2) \).

The correlation values \( \eta_{12} \) from Tables 3-5 were deduced by running the Monte Carlo algorithm \( \text{MCRCC} \) for samples having the volume \( n = 10^7 \).

**Table 3.** \( q_1 = q_2 = 0.5 \), \( n = 10^7 \) Monte Carlo simulations with \( \text{MCRCC} \)

| \( c_{12} \) | -0.999 | -0.9 | -0.8 | -0.7 | -0.6 | -0.5 | -0.4 |
| \( \eta_{12} \) | -0.9714 | -0.7129 | -0.5906 | -0.4938 | -0.4099 | -0.3335 | -0.2621 |
| \( c_{12} \) | -0.3 | -0.2 | -0.1 | 0 | 0.1 | 0.2 | 0.3 |
| \( \eta_{12} \) | -0.1940 | -0.1282 | -0.0637 | 0.0001 | 0.0638 | 0.1284 | 0.1943 |
| \( c_{12} \) | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.999 |
| \( \eta_{12} \) | 0.2622 | 0.3333 | 0.4096 | 0.4937 | 0.5904 | 0.7129 | 0.9714 |

**Table 4.** \( q_1 = 0.4 \), \( q_2 = 0.6 \), \( n = 10^7 \) simulations with \( \text{MCRCC} \)

| \( c_{12} \) | -0.999 | -0.9 | -0.8 | -0.7 | -0.6 | -0.5 | -0.4 |
| \( \eta_{12} \) | -0.9713 | -0.7106 | -0.5872 | -0.4902 | -0.4060 | -0.3298 | -0.2588 |
| \( c_{12} \) | -0.3 | -0.2 | -0.1 | 0 | 0.1 | 0.2 | 0.3 |
| \( \eta_{12} \) | -0.1912 | -0.1261 | -0.0628 | -0.0004 | 0.0616 | 0.1240 | 0.1869 |
| \( c_{12} \) | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.999 |
| \( \eta_{12} \) | 0.2512 | 0.3173 | 0.3861 | 0.4589 | 0.5364 | 0.6181 | 0.6667 |
### Table 5. \( q_1 = 0.5, q_2 = 0.7 \) , \( n = 10^7 \) simulations with MCRCC

<table>
<thead>
<tr>
<th>( c_{12} )</th>
<th>-0.999</th>
<th>-0.9</th>
<th>-0.8</th>
<th>-0.7</th>
<th>-0.6</th>
<th>-0.5</th>
<th>-0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_{12} )</td>
<td>-0.6546</td>
<td>-0.6091</td>
<td>-0.5293</td>
<td>-0.4529</td>
<td>-0.3809</td>
<td>-0.3125</td>
<td>-0.2472</td>
</tr>
</tbody>
</table>

Remark 8. The differences between the correlation values \( \eta_{12} = \text{Cor}(B_1, B_2) \) and \( c_{12} = \text{Cor}(X_1, X_2) \) are sometimes considerable. Graphic 2 gives us a suggestive illustration of this aspect (compare the differences between the continuous and dotted curves).

![Graphic 2](image)

**Graphic 2.** The ratio between the correlation indices \( \eta_{12} \) and \( c_{12} \)

Remark 9. We can use successively Proposition P4.5 and the discretization procedure DP to simulate directly samples from a tree type binary systems. See, for example, the one level tree system depicted in Figure 1, case 1.3.

6. Concluding remarks

We discussed two algorithms to generate random variates for a binary system \( B = (B_1, B_2, B_3, \ldots, B_k) \) with \( k \) components.

The algorithm GDRV uses as inputs all the probabilities \( p_{i_1, i_2, \ldots, i_k}, \quad i_1 \in \{0,1\}, \quad 1 \leq j \leq k \), which characterize the binary system \( B \). It is not so easy to apply practically the procedure GDRV for systems \( B \) which have a lot of components. In this case the quantity \( 2^k - 1 \) of the input data for GDRV algorithm becomes extremely large.
For this reason is suggested a new other algorithm based on the discretization procedure $DP$ to obtain arbitrary observations from $\mathcal{B}$. This procedure simulate better the real aspects. The correlation structure of a continuous system $X$ is inherited by the binary system $\mathcal{B}$ resulted after a discretization process. The relation between the correlation coefficients $c_{12} = Cor(X_1, X_2)$ and $\eta_{12} = Cor(B_1, B_2)$ can be determined by applying $MCRCC$ algorithm (see also Graphic 2).

References